TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 320, Number 2, August 1990

THE DIRICHLET PROBLEM FOR RADIALLY HOMOGENEOUS ELLIPTIC OPERATORS

RICHARD F. BASS

ABSTRACT. The Dirichlet problem in the unit ball is considered for the strictly elliptic operator $L = \sum a_{ij} D_{ij}$, where the a_{ij} are smooth away from the origin and radially homogeneous: $a_{ij}(rx) = a_{ij}(x)$, $r > 0$, $x \neq 0$. Existence and uniqueness are proved for solutions in a certain space of functions. Necessary and sufficient conditions are given for an extended maximum principle to hold.

1. INTRODUCTION

Let L be the second-order strictly elliptic operator defined by

$$Lf(x) = \sum_{i,j=1}^{d} a_{ij}(x) D_{ij} f(x),$$

where the a_{ij} are C^∞ on $\mathbf{R}^d - \{0\}$ and radially homogeneous:

$$a_{ij}(rx) = a_{ij}(x), \qquad x \neq 0, \ r > 0.$$

The main purpose of this paper is to investigate the Dirichlet problem in the unit ball for the operator L.

There is some question about what one means here by a solution to the Dirichlet problem. Of course, one does not expect the solution h necessarily to be in C^2; one usually wants h to be (locally) in the Sobolev space $W^{2,p}$ for some p. But a simple example of Pucci [11] shows that one cannot guarantee uniqueness of the solution, even with smooth boundary function, unless $p \geq d$, the dimension of the space. On the other hand, a recent example of Safonov [12] shows, among other things, that one may not have a solution at all unless $p \leq d/2$. In fact, there is an example by Lamberti and Manselli [10] of an operator (although not a radially homogeneous one in a ball) where for each p: either no solution exists in $W^{2,p}$ or else infinitely many solutions exist in $W^{2,p}$.

Received by the editors February 29, 1988 and, in revised form, October 10, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 60J60; Secondary 35J15.
Key words and phrases. Dirichlet problem, maximum principle, elliptic operator, radially homogeneous, diffusion processes, martingale problem.
Research supported in part by NSF grant DMS-8701073.

We formulate the Dirichlet problem slightly differently. Roughly speaking, we replace Lebesgue measure in the definition of $W^{2,p}$ with another measure that is naturally associated with the operator L. We consider the Dirichlet problem for boundary functions f that are bounded and continuous on ∂B_1, the boundary of the open unit ball B_1. First of all, we require of any candidate for solution h that

(1.1)
(i) h is bounded and continuous on $\overline{B}_1$ (closure of B_1);
(ii) h is C^2 in $B_1 - \{0\}$;
(iii) $Lh \equiv 0$ in $B_1 - \{0\}$; and
(iv) $h = f$ on ∂B_1.

We define $\Gamma(y)$ to be the Green function for L for the unit ball with pole at 0 (a precise meaning of this is given in §2). We then prove (Theorem 2.3) that there exists $\eta > 0$, depending only on the coefficients of a, with the property:

(1.2) There exists one and only one function h satisfying (1.1) and

$$\int_{B_R} \sum_{i,j=1}^{d} |D_{ij}h|^{1+\eta}(x)\Gamma(x)\,dx < \infty \quad \text{for all } R < 1.$$

Here B_R is the open ball of radius R about the origin.

Another main result of this paper concerns the extended maximal principle (cf. Gilbarg-Serrin [4]). We define a parameter $\overline{\mu}$ in terms of the a_{ij}. We then show (Theorem 2.1 and Proposition 2.2) that we have an extended maximum principle:

(1.3) $\sup_{x\in B_1-\{0\}} h(x) \le \sup_{x\in\partial B_1} h(x)$ whenever h is bounded and continuous on $\overline{B}_1 - \{0\}$ and $Lh \equiv 0$ on $B_1 - \{0\}$ if and only if $\overline{\mu} \ge 0$.

As a by-product of our methods, we obtain the estimates

$$(1.4) \qquad \nabla h \in L^{d+\varepsilon}(B_R), \qquad D_{ij}h \in L^{d/2+\varepsilon}(B_R) \quad \text{for all } R < 1.$$

The constant ε depends only on the coefficients of ellipticity, and ∇h denotes the gradient of h. It would be interesting to know if the estimates (1.4) hold for nonradially homogeneous operators as well. Also, is the formulation of the Dirichlet problem given above, using (1.1) and (1.2), applicable more generally?

The approach taken in this paper is probabilistic, and one of our motivations was to compare solving the Dirichlet problem for L to solving the corresponding martingale problem of probability theory. For the latter, the key step (see [1]) is to show that the largest eigenvalue α of a certain positive operator Q is simple. For the Dirichlet problem more is required: we must also estimate hitting probabilities, Green functions, and rate of growth of solutions to $Lh = 0$ in terms of α.

Concerning uniqueness (but not existence), there is a recent result of Caffarelli [15] that should be mentioned. Suppose the a_{ij} are smooth except at 0, strictly elliptic, but not necessarily radially homogeneous. Let a_{ij}^n be smooth approximations to the a_{ij} and let h_n be the solution to the corresponding Dirichlet problem. Then Caffarelli showed that the functions h_n converge, and the limit is independent of how the a_{ij} were smoothed.

In §2 we give some preliminaries and state our results precisely. In §3 we give a criterion for whether the Markov process associated to L ever hits the origin (nonpolar) or not (polar) and prove the extended maximum principle as a corollary. We then consider the more difficult of the two cases, the nonpolar one. In §4 we estimate hitting probabilities and the Green function, in §5 we establish existence of a solution to the Dirichlet problem, and in §6 we prove uniqueness of this solution. §7 covers the case where the origin is a polar set and the Markov process is transient, while §8 deals with the case where the origin is polar but the Markov process is neighborhood recurrent.

A few words about notation: We will use the letter c, with or without subscripts, to denote constants whose value is unimportant and may change from line to line. $B_r(x)$ denotes the open ball of radius r about x, B_r the open ball of radius r about 0.

2. Preliminaries and statement of results

Let the operator L be defined by

$$Lf(x) = \sum_{i,j=1}^{d} a_{ij}(x) D_{ij} f(x), \tag{2.1}$$

where the a_{ij} satisfy

(i) (strict ellipticity) there exists $\kappa > 0$ such that for all $(y_1, \ldots, y_d) \in \mathbf{R}^d$, $x \in \mathbf{R}^d$,

$$\kappa \sum_{i=1}^{d} y_i^2 \le \sum_{i,j=1}^{d} y_i y_j a_{ij}(x) \le \kappa^{-1} \sum_{i=1}^{d} y_i^2; \tag{2.2}$$

(ii) (smoothness) each a_{ij} is C^∞ on $\mathbf{R}^d - \{0\}$; and

(iii) (radial homogeneity) for each i, j, $a_{ij}(rx) = a_{ij}(x)$ whenever $r > 0$, $x \neq 0$.

Let S be the unit sphere. If we write L in terms of polar coordinates (r, θ), $r > 0$, $\theta \in S$, we get

$$\begin{aligned} Lf(r,\theta) = {} & \frac{1}{2}\gamma(r,\theta) f_{rr}(r,\theta) + \gamma(r,\theta)\frac{2\mu(r,\theta)+1}{2r} f_r(r,\theta) \\ & + \frac{\gamma(r,\theta)}{r^2} L_S f(r,\theta) + \frac{\gamma(r,\theta)}{r} M f(r,\theta), \end{aligned}$$

where f_r, f_{rr} are the derivatives of f in the radial direction, $x = (x_1, \dots, x_d) = (r, \theta)$,

$$\gamma(r, \theta) = 2r^{-2} \sum_{i,j=1}^{d} x_i a_{ij}(x) x_j,$$

$$\mu(r, \theta) = \operatorname{trace}(a)/\gamma(r, \theta) - 1,$$

L_S is an elliptic operator on S containing the f_θ and $f_{\theta\theta}$ terms, and M is an operator containing the mixed partials $f_{r\theta}$. The reason for the strange form of the coefficient of f_r will be apparent shortly. By (2.2)(iii), γ, μ, and the coefficients of L_S and M are independent of r, and we will write $\gamma(\theta)$, $\mu(\theta)$. By (2.2)(i), γ is bounded above and below away from 0. Hence using (2.2)(ii), γ, μ, and the coefficients of L_S and M are C^∞, and L_S is strictly elliptic.

There is a unique strong Markov process (P^θ, θ_t) with state space S and infinitesimal generator L_S (see [13]). Since S is a compact manifold, (P^θ, θ_t) has an invariant probability measure $\nu(d\theta)$ on S [3, Example 3.1]. Equivalently, let $\nu(d\theta)$ be the measure with $\nu(S) = 1$ whose density $v(\theta)$ with respect to surface measure is the nonnegative solution to $L_S^* v = 0$, where L_S^* is the adjoint operator to L_S.

Define

$$\overline{\mu} = \int \mu(\theta)\nu(d\theta). \tag{2.3}$$

We then have the extended maximum principle:

Theorem 2.1. *Suppose* $\overline{\mu} \geq 0$. *Suppose* h *is bounded and continuous on* $\overline{B}_1 - \{0\}$, C^2 *in* $B_1 - \{0\}$, *and satisfies* $Lh \equiv 0$ *in* $B_1 - \{0\}$. *Then*

$$\sup_{x \in B_1 - \{0\}} h(x) \leq \sup_{x \in \partial B_1} h(x).$$

This is complemented by

Proposition 2.2. *If* $\overline{\mu} < 0$, *there exists* h *bounded by* 0 *and* 1, *continuous on* $\overline{B}_1$, C^2 *in* $B_1 - \{0\}$, *and* $Lh \equiv 0$ *in* $B_1 - \{0\}$ *such that* $h(0) = 1$ *and* $h \equiv 0$ *on* ∂B_1.

Theorem 2.1 and Proposition 2.2 are proved in §3.

Consider the martingale problem for L starting at x. This is the question of existence and uniqueness of a probability P^x on the space of continuous paths in $\mathbf{R}^d$ such that

$$P^x(X_0 = x) = 1 \tag{2.4}$$

and

(2.5)

$$f(X_t) - f(X_0) - \int_0^t Lf(X_s)\,ds \quad \text{is a } P^x\text{-local martingale for each } f \in C^2.$$

Existence follows by [13, Exercise 14.4.3]. Since the a_{ij} are C^∞ on $\mathbf{R}^d - \{0\}$, the proof that P^x is determined up to the first time the process hits 0 is standard (see [13]). Uniqueness for the martingale problem then follows by [1] (although the context is slightly different, the proofs of [1, §5] apply verbatim).

By Krylov [7], there exists a function $\Gamma(x)$, the Green function, such that

$$E^0 \int_0^{\tau_1} 1_A(X_s)\, ds = \int_A \Gamma(x)\, dx \quad \text{for all } A \subseteq B_1, \tag{2.6}$$

where

$$\tau_r = \inf\{t \colon |X_t| = r\}. \tag{2.7}$$

One can show that Γ is continuous away from 0, but we do not need this.

Our formulation and solution to the Dirichlet problem is then given by

Theorem2.3. *There exists $\eta > 0$, depending only on κ, such that if f is continuous on ∂B_1, then there exists one and only one function h satisfying*

(2.8)
- (i) *h is bounded and continuous on $\overline{B}_1$;*
- (ii) *h is C^2 on $B_1 - \{0\}$;*
- (iii) *$Lh = 0$ on $B_1 - \{0\}$;*
- (iv) *h agrees with f on ∂B_1; and*
- (v) *$\int_{B_R} \sum_{i,j=1}^d |D_{ij}h|^{1+\eta}(x)\Gamma(x)\, dx < \infty$ for all $R < 1$.*

For the case when $\overline{\mu} \geq 0$, it will turn out that (i)–(iv) are sufficient for uniqueness.

As a by-product of our proof of Theorem 2.3, we will get

Theorem 2.4. *There exists $\varepsilon > 0$, depending only on κ, such that if h satisfies (2.8)(i)–(v), then*

$$\nabla h \in L^{d+\varepsilon}(B_R), \qquad D_{ij}h \in L^{d/2+\varepsilon}(B_R) \quad \textit{for all } R < 1. \tag{2.9}$$

Theorems 2.3 and 2.4 will be proved in §5 (existence when $\overline{\mu} < 0$), §6 (uniqueness when $\overline{\mu} < 0$), §7 (the case $\overline{\mu} > 0$), and §8 (the case $\overline{\mu} = 0$).

Let $\widetilde{L} = \gamma(\theta)^{-1}L$. Note that $Lh = 0$ if and only if $\widetilde{L}h = 0$. The Markov process corresponding to $\widetilde{L}$ may be obtained from that of L by a time change, and it is well known (cf. [13, §6.5]) that if $\widetilde{\Gamma}$ is the corresponding Green function, $\widetilde{\Gamma}(x) = \gamma(\theta)\Gamma(x)$. Recall that γ is bounded above and bounded below away from 0. Thus, in the statements of Theorems 2.1, 2.3, and 2.4, there is no loss of generality in replacing L by $\widetilde{L}$ or, equivalently, in assuming

Assumption 2.5. $\sum_{i,j=1}^d x_i a_{ij}(x) x_j / |x|^2 = \frac{1}{2}$ *for all* x.

Assumption 2.5 will remain in force for the remainder of the paper.

Writing L in polar coordinates, we now get the much simpler expression

$$Lf(r,\theta) = \frac{1}{2} f_{rr}(r,\theta) + \frac{2\mu(\theta)+1}{2r} f_r(r,\theta) + \frac{1}{r^2} L_S f(r,\theta) + \frac{1}{r} Mf(r,\theta), \tag{2.10}$$

where

$$\mu(\theta) = \text{trace } a(x) - 1, \quad x = (r, \theta).$$

We will use the following two lemmas frequently.

Lemma 2.6. *Suppose D is an open region in $\mathbf{R}^d$, $T = \inf\{t: X_t \notin D\}$, and f is bounded on ∂D. Then $h(x) = E^x f(X_T)$ is continuous in D.*

Proof. This follows by the proof of Theorem 2 in [8]. □

Lemma 2.7. *Suppose D, T, f, and h are as in Lemma 2.6. If $0 \notin D$, then h is C^∞ in D and $Lh = 0$ there.*

Proof. Fix $x_0 \in D$ and choose ε small enough so that $B_{2\varepsilon}(x_0)$, the ball of radius 2ε about x_0, lies in D. Let $S_r = \inf\{t: X_t \notin B_r(x_0)\}$. By the strong Markov property, $h(x) = E^x h(X_{S_\varepsilon})$ if $x \in B_\varepsilon(x_0)$.

By Lemma 2.6, h is continuous in $\overline{B_\varepsilon(x_0)}$. Let $u(x)$ be the solution to the Dirichlet problem for L in $B_\varepsilon(x_0)$ with boundary function h. Since the a_{ij} are uniformly C^∞ on $B_{2\varepsilon}(x_0)$, it is well known [5, Chapter 6] that u is C^∞ in $B_\varepsilon(x_0)$, continuous on $\overline{B_\varepsilon(x_0)}$, and $Lu = 0$ in $B_\varepsilon(x_0)$. By Ito's formula, $u(X_{t\wedge S_r})$ is a martingale for $r < \varepsilon$. So

$$u(x) = E^x u(X_{t\wedge S_r}), \qquad r < \varepsilon.$$

Using the boundedness of u on $\overline{B_\varepsilon(x_0)}$, let $t \to \infty$, then $r \to \varepsilon$ to get

$$u(x) = E^x u(X_{S_\varepsilon}) = E^x h(X_{S_\varepsilon}) = h(x).$$

Therefore $h(x)$ is C^∞ and $Lh = 0$ in the interior of $B_\varepsilon(x_0)$. Since x_0 was arbitrary, this completes the proof. □

3. Polar or nonpolar

Let (P^x, X_t) be the unique solution to the martingale problem for L starting at x. We will use repeatedly the fact that (P^x, X_t) forms a strong Markov process [13, §6.2].

In this section we give a criterion in terms of $\overline{\mu}$ for whether X_t hits the origin in finite time or not. We use this to prove the extended maximum principle.

We begin by expressing (P^x, X_t) in polar coordinates (cf. the skew product decomposition of [14]). Fix $x = (r, \theta)$, and write $X_t = (\widetilde{R}_t, \widetilde{\theta}_t)$. Recall that if $\sigma_{ij}(x)$ is a square root of $a_{ij}(x)$, we can write

$$X_t^{(i)} = x^{(i)} + \int_0^t \sum_{j=1}^d \sigma_{ij}(X_s)\, d\widehat{W}_x^{(s)}, \qquad i = 1, \dots, d, \tag{3.1}$$

for some d-dimensional Brownian motion in $\widehat{W}$ (see [13]).

It follows by Ito's lemma, then, that $\widetilde{R}_t$ solves

$$\widetilde{R}_t = r + \widetilde{W}_t + \int_0^t (2\mu(\tilde{\theta}_s) + 1)/2R_s\, ds \tag{3.2}$$

up until the first time $|X_t| = 0$ and that $(P^{(r,\theta)}, \tilde{\theta}_t)$ solves the martingale problem for $r^{-2}L_S$ starting at (r, θ). Here $\widetilde{W}_t$ is a standard one-dimensional Brownian motion.

Now let (R_t, θ_t) be the time change of $(\widetilde{R}_t, \tilde{\theta}_t)$ defined by

$$R_t = \widetilde{R}(B_t^{-1}), \qquad \theta_t = \tilde{\theta}(B_t^{-1}),$$

where

$$B_t = \int_0^t \widetilde{R}_s^2 \, ds.$$

Then, up until the first time of hitting 0, R_t solves

$$dR_t = R_t \, dW_t + (\mu(\theta_t) + \tfrac{1}{2}) R_t \, dt, \qquad R_0 = r, \tag{3.3}$$

where W_t is a standard Brownian motion.

Let

$$A_t = \int_0^t \mu(\theta_t) \, ds. \tag{3.4}$$

The equation (3.3) is linear in R, hence

$$R_t = r \exp(W_t + A_t). \tag{3.5}$$

Also, note that $(P^{(r,\theta)}, \theta_t)$ solves the martingale problem for L_S on S starting at θ. Since L_S is smooth and strictly elliptic, S is a smooth manifold, and the coefficients of L_S depend only on θ, there is at most one solution to the martingale problem for L_S on S starting at θ. We denote it by P^θ. By [13], (P^θ, θ_t) forms a strong Markov process with state space S.

Theorem 3.1. *For each* θ,

(a) *if* $\overline{\mu} > 0$, $W_t + A_t \to +\infty$, P^θ*-a.s.*;

(b) *if* $\overline{\mu} < 0$, $W_t + A_t \to -\infty$, P^θ*-a.s.*

Proof. A_t is an additive functional of θ_t, and by the ergodic theorem (see [2] or [3]), $A_t/t \to \overline{\mu}$, P^θ-a.s.

Suppose $\overline{\mu} > 0$. If $c > 0$, it is well known that $W_t + ct \to +\infty$, a.s. as $t \to \infty$. Given ε, there exists t_0 such that

$$P^\theta(A_t \leq \tfrac{1}{2}\overline{\mu} t \text{ for some } t > t_0) < \varepsilon.$$

Then for each N

$$P^\theta(W_t + A_t \geq N \text{ eventually }) \geq P^\theta(W_t + \tfrac{1}{2}\overline{\mu} t \geq N \text{ eventually }) - \varepsilon \geq 1 - \varepsilon,$$

which proves (a).

The proof of (b) is similar. □

Let

$$\tau_r = \inf\{t \colon |X_t| = r\}. \tag{3.6}$$

We use Theorem 3.1 to get the criterion:

Theorem 3.2. *Suppose* $x \neq 0$.

(a) *If* $\overline{\mu} > 0$, $P^x(\tau_0 < \infty) = 0$ *and* $P^x(|X_t| \to \infty$ *as* $t \to \infty) = 1$;

(b) *if* $\overline{\mu} < 0$, $P^x(\tau_0 < \infty) = 0$ *and* $P^x(|X_t| \to \infty$ *as* $t \to \infty) = 0$;

(c) *if* $\overline{\mu} = 0$, $P^x(\tau_0 < \infty) = 0$, $P^x(|X_t| \to \infty) = 0$, *for all* r, $P^x(\tau_r < \infty) = 1$.

Proof. Write $x = (r, \theta)$. If $\overline{\mu} > 0$, then $W_t + A_t \to +\infty$, a.s. by Theorem 3.1. Since $\log|R_t| = \log r + (W_t + A_t)$, then $\inf_t \log|R_t| > -\infty$, a.s. so $\tau_0 = \infty$, a.s. and $\liminf_{t\to\infty} \log|R_t| = +\infty$, or $|R_t| \to \infty$ as $t \to \infty$. Part (a) follows easily.

Suppose now $\overline{\mu} < 0$, and let $\varepsilon > 0$. Since $W_t + A_t \to -\infty$, a.s., we can find N_0 such that for all M

$$P^\theta(\log r + W_t + A_t \text{ hits } -M \text{ before hitting } N) \geq 1 - \varepsilon$$

whenever $N \geq N_0$. Since the time change that gives $|X_t| = \widetilde{R}_t$ from R_t is nondegenerate (i.e., dB_t/dt is bounded below) when $R_t \geq e^{-M} > 0$, we have

$$P^x(|X_t| \text{ hits } e^{-M} \text{ before hitting } e^N) \geq 1 - \varepsilon.$$

Let $M \to \infty$ to get $P^x(\tau_0 < \tau_{\exp(N)}) \geq 1 - \varepsilon$, then let $N \to \infty$, then $\varepsilon \to 0$. The second part of (b) follows from the first part and an elementary renewal argument.

Recall that (P^θ, θ_t) is the process on S with generator L_S. Since

$$\int_S \mu(\theta)\nu(d\theta) = \overline{\mu} = 0,$$

it is well known (see [2]) that $E^\theta \mu(\theta_t)$ goes to 0 exponentially fast, uniformly in θ, as $t \to \infty$.

Let

$$d(\theta) = \int_0^\infty E^\theta \mu(\theta_t)\, dt. \tag{3.7}$$

Since $\mu(\theta)$ is C^∞ and the coefficients of L_S are C^∞ and strictly elliptic, $d(\theta)$ is C^∞ and

$$L_S d = -\mu. \tag{3.8}$$

Set

$$u(x) = \log r + d(\theta), \qquad x = (r, \theta). \tag{3.9}$$

Then applying (2.10), noting that the $u_{r\theta}$ terms are 0, and using (3.8),

$$Lu(x) = 0, \qquad x \neq 0.$$

Hence $U_t = u(X_{t\wedge\tau_0})$ is a local martingale.

By [1, Lemma 5.1], $P^x(\tau_r < \infty) = 1$ if $|x| < r$. So U_t is certain to exit the interval $[-M, N]$. Then, since U_t is a time change of Brownian motion,

$$P^x(U_t \text{ hits } -M \text{ before hitting } N) = (N - u(x))/(M + N).$$

Using the definition of u, noting that $d(\theta)$ is bounded, and letting $M \to \infty$ leads as above to $P^x(\tau_0 < \infty) = 0$.

Similarly, holding M fixed and letting $N \to \infty$ leads to $P^x(\tau_r < \infty) = 1$ for all $r > 0$ as above.

A renewal argument shows that $P^x(|X_t| \le r \text{ i.o. as } t \to \infty) = 1$, which implies $P^x(|X_t| \to \infty) = 0$. □

We refer to the cases where $\overline{\mu} \ge 0$ as polar since 0 is a polar set, the case $\overline{\mu} < 0$ as nonpolar.

The proof of Theorem 2.1 is now easy.

Proof of Theorem 2.1. Suppose $\overline{\mu} \ge 0$, and suppose h is bounded by M on $\overline{B}_1 - \{0\}$. Let $\gamma \in (0, 1)$. By Lemma 2.7, the solution to the Dirichlet problem in $B_1 - B_\gamma$ with boundary values h is again h, and if $|\gamma| < x$,

$$\begin{aligned} h(x) &= E^x h(X_{\tau_1 \wedge \tau_\gamma}) \\ &= E^x(h(X_{\tau_1}); \tau_1 < \tau_\gamma) + E^x(h(X_{\tau_\gamma}); \tau_\gamma < \tau_1) \\ &\le \sup_{y \in \partial B_1} h(y) + MP^x(\tau_\gamma < \tau_1). \end{aligned}$$

Let $\gamma \to 0$; by the continuity of paths, $\lim_{\gamma \to 0} P^x(\tau_\gamma < \tau_1) = P^x(\tau_0 < \tau_1) = 0$, using Theorem 3.2. □

Proof of Proposition 2.2. Let $h(x) = P^x(\tau_0 < \tau_1)$. By Lemmas 2.6 and 2.7, h is C^2 in $B_1 - \{0\}$ and $Lh = 0$ there. (Here $D = B_1 - \{0\}$, $f(0) = 1$, $f \equiv 0$ on ∂B_1.) It remains only to show (i) $\lim_{x \to 0} h(x) = 1$ and (ii) $\lim_{x \to y} h(x) = 0$ if $y \in \partial B_1$.

The first follows from scaling (see [1, proof of Proposition 5.3]): if $x = \varepsilon y$ where $|y| = 1$, then

$$P^x(\tau_0 < \tau_1) = P^y(\tau_0 < \tau_{1/\varepsilon}) \to P^y(\tau_0 < \tau_\infty) = 1$$

as $\varepsilon \to 0$.

For the second, by the strong Markov property, $h(x) = E^x(h(X_{\tau_\gamma}); \tau_\gamma < \tau_1)$, where $\gamma = 1/2 < |x|$. The coefficients of L are smooth on $B_1 - B_\gamma$. So the solution to the classical Dirichlet problem for L with boundary values h on ∂B_γ and 0 on ∂B_1 is continuous up to and including the boundary [5, Chapter 6]. By Lemma 2.7, the solution is $E^x(h(X_{\tau_\gamma}); \tau_\gamma < \tau_1) + E^x(0; \tau_1 < \tau_\gamma) = h(x)$. Hence h is continuous on ∂B_1. □

4. Green function—nonpolar case

In the next three sections we consider the nonpolar case: $\overline{\mu} < 0$, which is the most difficult. First we introduce a positive operator Q, examine the eigenvalues of Q, and then estimate the probability of hitting 0 and estimate the Green function with pole at 0 in terms of the largest eigenvalue of Q.

We suppose throughout this section that $\overline{\mu} < 0$.

For $b \in (0, 1)$, let Q_b be the operator on functions on S defined by

$$Q_b f(x) = E^{bx}(f(X_{\tau_1}); \tau_1 < \tau_0), \qquad x \in S, \tag{4.1}$$

where τ_0, τ_1 are defined by (3.6). We write simply Q for $Q_{1/2}$.

Proposition 4.1. *Let $b \in (0, 1)$. Fix $x_0 \in S$, and let $\pi(dy) = Q_b(x_0, dy) = P^{bx_0}(X_{\tau_1} \in dy; \tau_1 < \tau_0)$. Then*

(a) *Q_b is a bounded operator on $L^2(d\pi)$;*

(b) *Q_b is a bounded operator from $L^1(d\pi)$ to $\mathscr{C}(S)$, the continuous functions on S;*

(c) *the largest eigenvalue λ_b of Q_b is positive and strictly larger in absolute value than any other eigenvalue of Q_b, $\lambda_b < 1$, and the corresponding eigenfunction e_b is continuous and strictly positive.*

Proof. By the Harnack inequality of Krylov-Safonov [9], there exists a constant θ depending only on κ of (2.2) and b such that if $f: S \to [0, \infty)$,

$$\theta \le \frac{E^{bx}(f(X_{\tau_1}); \tau_1 < \tau_0)}{E^{bx_0}(f(X_{\tau_1}); \tau_1 < \tau_0)} \le \theta^{-1} \quad \text{for all } x \in S. \tag{4.2}$$

(Recall Lemma 2.7.) Consequently, $Q_b(x, dy) << \pi(dy)$, and the density $q_b(x, y) = Q_b(x, dy)/\pi(dy)$ may be taken to be bounded by θ^{-1}.

Since

$$Q_b f(x) = \int f(y) Q_b(x, dy) = \int f(y) q_b(x, y) \pi(dy)$$

and

$$\int_S \int_S q_b(x, y)^2 \pi(dx) \pi(dy) \le \theta^{-2},$$

(a) follows by Cauchy-Schwarz.

To prove (b), suppose $f \in L^1(d\pi)$ and $f \ge 0$. Then $Q_b f(x_0) \le \|f\|_{L^1(d\pi)}$. By (4.2), $Q_b f(x) \le \theta^{-1} Q_b f(x_0)$ for all $x \in S$, which proves $Q_b f$ is bounded on S.

By the Harnack inequality of Krylov-Safonov again, if $x \in S$ and ε is taken small enough so that $B_{2\varepsilon}(bx) \subseteq B_1(0)$, there exists a constant c (depending on ε and κ) such that $h(y) = E^y(f(X_{\tau_1}); \tau_1 < \tau_0) \le c Q_b f(x)$ for $y \in B_\varepsilon(bx)$. By the strong Markov property,

$$Q_b f(x) = E^{bx} h(X_{S_\varepsilon}),$$

where

$$S_\varepsilon = \inf\{t: X_t \notin B_\varepsilon(x_0)\}.$$

So by Lemma 2.6, $Q_b f$ is continuous.

The operator Q_b is strongly positive and compact; for a proof see [1, §5]. Therefore, just as in the argument there, assertion (c), except for the $\lambda_b < 1$ claim, follows by a the Krein-Rutman theorem.

Pick x such that $e_b(x) \equiv \sup_y e_b(y)$. Then

$$\lambda_b e_b(x) = Q_b e_b(x) = E^{bx}(e_b(X_{\tau_1}); \tau_1 < \tau_0) \le \sup_y e_b(y) P^{bx}(\tau_1 < \tau_0)$$
$$= e_b(x) P^{bx}(\tau_1 < \tau_0).$$

Since e_b is strictly positive and $P^{bx}(\tau_1 < \tau_0) < 1$ by the proof of Theorem 3.3, $\lambda_b \le P^{bx}(\tau_1 < \tau_0) < 1$. □

We normalize e_b so that $\int_S e_b^2 \, d\pi = 1$.

Although we know $\lambda_b > 0$, we need for use in §5 to show that it is greater than a constant depending only on κ and b.

Corollary 4.2. *There exists a constant c depending only on κ and b such that $\lambda_b \ge c$.*

Proof. Analogously to the last paragraph of the proof of Proposition 4.1, $\lambda_b \ge \inf_{x \in S} P^{bx}(\tau_1 < \tau_0)$. But $P^{bx}(\tau_1 < \tau_0)$ can be bounded below by a constant depending only on κ and b by the Stroock-Varadhan support theorem [6, Exercise 6.7.5] (cf. [1, proof of Theorem 5.4]). □

Let

$$d(\theta) = e_{1/2}(\theta), \qquad \theta \in S, \tag{4.3}$$

$$\alpha = \log \lambda_{1/2} / \log(1/2) > 0, \tag{4.4}$$

and let

$$u(x) = r^\alpha d(\theta), \qquad x = (r, \theta). \tag{4.5}$$

Proposition 4.3. (a) $u(x) = E^x(d(X_{\tau_1}); \tau_1 < \tau_0)$, $x \in B_1$;

(b) $u(X_{t \wedge \tau_0})$ *is a P^x-local martingale for all x;*

(c) d *is bounded above and bounded below away from* 0; d *is C^∞ on S.*

Proof. Suppose $b \in (0, 1)$. By the strong Markov property and radial homogeneity, $(Q_b)^n = Q_{b^n}$, and so $Q_{b^n} e_b = \lambda_b^n e_b$. By the Krein-Rutman theorem [6, Theorem 6.3], Q_{b^n} has only one strictly positive eigenfunction, hence e_{b^n} is a constant multiple of e_b, and $\lambda_{b^n} = (\lambda_b)^n$.

Now suppose $|x| = (1/2)^{m/n}$, m, n positive integers, and let $b = |x|$. $Q_{b^n} = Q_{(1/2)^m}$, hence $\lambda_b^n = \lambda_{1/2}^m$, and $Q_b d = \lambda_b d$. Therefore

$$E^x(d(X_{\tau_1}); \tau_0 < \tau_0) = Q_b d(x/b) = \lambda_b d(x/b) = (\lambda_{1/2})^{m/n} d(x/|x|)$$
$$= |x|^\alpha d(x/|x|).$$

This proves (a) for $|x|$ of the form $(1/2)^s$, s rational. $E^x(d(X_{\tau_1}); \tau_1 < \tau_0)$ is continuous in $B_1 - \{0\}$ by Lemma 2.6, and u is evidently continuous as a function of $|x|$. So (a) holds for $x \in B_1 - \{0\}$.

Now

$$|E^x(d(X_{\tau_1}); \tau_1 < \tau_0)| \le \|d\| P^x(\tau_1 < \tau_0) \to 0 \quad \text{as } x \to 0$$

by the argument in the proof of Proposition 2.2. So (a) holds for $x = 0$.

By Lemma 2.7, $u(x)$ is C^∞ in $B_1 - \{0\}$ and $Lu = 0$ there. So

$$0 = Lu = r^{\alpha-2}[\alpha(\alpha-1)d/2 + \alpha(2\mu+1)d/2 + L_S d + \alpha M(rd)] \tag{4.6}$$

for $r \in (0, 1)$. But then the expression inside the brackets, which is independent of r, is 0 for all θ, hence (4.6) holds for all $r > 0$. (b) then follows by Ito's lemma.

We have already mentioned that u is C^∞ in $B_1 - \{0\}$. This shows d is smooth. The boundedness part of (c) follows immediately from Proposition 4.1(c). □

Corollary 4.4. *$d(\theta)$ is bounded above and below by constants depending only on κ. There is a modulus of continuity for $d(\theta)$ depending only on κ.*

Proof. Since $d = e_{1/2}$ is normalized to have L^2 norm one with respect to a certain measure, there is at least one point x for which $d(x) \le 1$. Hence $Q\,d(x) \le \lambda_{1/2}$, and by (4.2),

$$\sup_{y\in S} d(y) = (\lambda_{1/2})^{-1} \sup_{y \in S} Q\,d(y) \le (\lambda_{1/2})^{-1}\theta^{-1} Q\,d(x) \le \theta^{-1}.$$

Since there must be at least one point at which d is ≥ 1, the lower bound is similar. The assertion about the modulus of continuity follows from Lemma 2.6 and its proof. □

The $Q = Q_{1/2}$ that we have defined here is, by scaling, the same as the Q defined in [1]. Williams [14] has given a quite different method of constructing a positive solution to $Lu = 0$. Our construction sheds additional light on hers, and vice versa.

Having constructed the function u, we can now begin doing some estimation. Let

$$V_i = \{x : u(x) \le 2^i\}, \qquad i = \ldots, -2, -1, 0, 1, 2, \ldots, \tag{4.7}$$

$$F_i = \partial V_i, \tag{4.8}$$

and

$$\sigma_i = \inf\{t : X_t \in F_i\}. \tag{4.9}$$

Proposition 4.5. *There exist constants c_1, $c_2 > 0$ such that*

$$c_1|x|^\alpha \le P^x(\tau_1 < \tau_0) \le c_2|x|^\alpha$$

for x sufficiently small.

Proof. Since $d(\theta)$ is bounded below there exists i_2 such that $V_{i_2} \supseteq B_1$. Then

$$\begin{aligned} P^x(\tau_1 < \tau_0) \ge P^x(\sigma_{i_2} < \tau_0) &= P^x(u(X_t) \text{ hits } 2^{i_2} \text{ before hitting } 0) \\ &= u(x)/2^{i_2} \ge c_1|x|^\alpha, \end{aligned}$$

since $u(X_t)$ is a martingale.

Similarly, there exists c_1 such that $V_{i_1} \subseteq B_1$, and then

$$P^x(\tau_1 < \tau_0) \le P^x(\sigma_{i_1} < \tau_0) = P^x(u(X_t) \text{ hits } 2^{i_1} \text{ before hitting } 0)$$
$$= u(x)/2^{i_1} \le c_2|x|^\alpha. \quad \square$$

As a corollary, we have

$$\lim_{|x|\to 0}(\log P^x(\tau_1 < \tau_0)/\log|x|) = \alpha. \tag{4.10}$$

On the other hand, it is possible to estimate $P^x(\tau_1 < \tau_0)$ in terms of the probability that $\log|x|+W_t+A_t$ ever hits 0 (see §3). Although this does not quite fit into standard large-deviations theory, one might be able to get asymptotic estimates by techniques from that theory; tying $\mu(\theta)$ and α together would answer some questions raised by Williams [14] in a similar context.

Lemma 4.6. $\alpha \le 1$.

Proof. From (2.10) and Assumption 2.5, $\mu(\theta) \ge -\frac{1}{2}$.

From (2.10) and Ito's formula, $|X_t|$ is a submartingale whose martingale part is a Brownian motion, $\widehat{W}_t$. Hence, for x small,

$$P^x(\tau_1 < \tau_0) \ge P^0(\widehat{W}_t + |x| \text{ hits } 1 \text{ before hitting } 0) = |x|.$$

By Proposition 4.5,

$$c_2|x|^\alpha \ge |x|$$

for all $|x|$ sufficiently small, which implies $\alpha \le 1$. $\square$

We now come to the main estimate of this section.

Theorem 4.7. *There exist constants c_1 and c_2 such that*

$$\Gamma(x) \le c_2|x|^{2-d-\alpha} \quad \text{a.e. on } B_1$$

and

$$\Gamma(x) \ge c_1|x|^{2-d-\alpha} \quad \text{a.e. on some neighborhood of } 0.$$

Proof. Let

$$\begin{aligned} A_i &= V_{i+4} - V_{i-3}, \\ G_i &= \sup\{g_{A_i}(x, y) : x \in F_{i-2} \cup F_{i+3},\ y \in V_{i+2} - V_{i-1}\}, \end{aligned} \tag{4.11}$$

where g_{A_i} is the Green function for A_i with pole at x. Since L is smooth on V_{i-3}^c, g_{A_i} is continuous in x and y except at $x = y$. Hence $G_i < \infty$.

Suppose $y \in V_{i+2} - V_{i-1}$ and let $B_\varepsilon(y)$ be the ball of radius ε about y, ε sufficiently small so that $B_{2\varepsilon}(y) \subseteq V_{i+3} - V_{i-1}$. We will write $g_{A_i}(x, B_\varepsilon(y))$ for $\int_{B_\varepsilon(y)} g_{A_i}(x, z)\,dz$. Using scaling with the factor $2^{-1/\alpha}$ (cf. [1, Proposition 5.3]), the expected amount of time spent in $B_\varepsilon(y)$ before leaving A_i starting at

x is $2^{-2/\alpha}$ times the expected amount of time spent in $B_{2^{1/\alpha}\varepsilon}(2^{1/\alpha}y)$ starting at $2^{1/\alpha}x$ before leaving A_{i+1}, or

$$g_{A_i}(x, B_\varepsilon(y)) = 2^{-2/\alpha} g_{A_{i+1}}(2^{1/\alpha}x, B_{2^{1/\alpha}\varepsilon}(2^{1/\alpha}y)).$$

Dividing by the volume of $B_\varepsilon(y)$, letting $\varepsilon \to 0$, and using the continuity at y,

$$g_{A_i}(x, y) = 2^{-2/\alpha} 2^{d/\alpha} g_{A_{i+1}}(2^{1/\alpha}x, 2^{1/\alpha}y). \tag{4.12}$$

Hence

$$G_i = 2^{(d-2)/\alpha} G_{i+1}. \tag{4.13}$$

Fixing i_2 so that $B_1 \subseteq V_{i_2-5}$ and then induction gives

$$G_i = 2^{(d-2)(i_2-i)/\alpha} G_{i_2} \le c 2^{(2-d)i/\alpha}. \tag{4.14}$$

Now let

$$H_i^\varepsilon = \sup\{g(x, B_\varepsilon(y)) \colon x \in F_{i-2} \cup F_{i+3}, y \in V_{i+1} - V_i\}, \tag{4.15}$$

where

$$g(x, B_\varepsilon(y)) = E^x \int_0^{\sigma_{i_2}} 1_{B_\varepsilon(y)}(X_s)\, ds,$$

σ_{i_2} is defined by (4.9), and $B_{2\varepsilon}(y) \subseteq V_{i+2} - V_{i-1}$ for all $y \in V_{i+1} - V_i$.

Fix $y \in V_{i+1} - V_i$, $i < i_2 - 5$. Starting at $z \in F_{i-3}$, one is certain to hit F_{i-2} before entering $B_\varepsilon(y)$, and so by the strong Markov property, $g(z, B_\varepsilon(y)) \le H_i^\varepsilon$.

If $z \in F_{i+4}$,

$$\begin{aligned} P^z(\sigma_{i+3} < \sigma_{i_2}) &= P^z(u(X_t) \text{ hits } 2^{i+3} \text{ before hitting } 2^{i_2}) \\ &= (2^{i_2} - 2^{i+4})/(2^{i_2} - 2^{i+3}) \le 1 - c2^i. \end{aligned} \tag{4.16}$$

So by the strong Markov property, $g(z, B_\varepsilon(y)) \le (1 - c2^i)H_i^\varepsilon$ if $z \in F_{i+4}$.

Let $T_i = \inf\{t \colon X_t \notin A_i\}$. Then just as in (4.16),

$$P^x(X_{T_i} \in F_{i+4}) = \begin{cases} (2^7 - 1)^{-1}, & x \in F_{i-2}, \\ (2^6 - 1)/(2^7 - 1), & x \in F_{i+3}. \end{cases} \tag{4.17}$$

Using the strong Markov property again and (4.17), if $x \in F_{i-2} \cup F_{i+3}$,

$$\begin{aligned} g(x, B_\varepsilon(y)) &= g_{A_i}(x, B_\varepsilon(y)) + E^x g(X_{T_i}, B_\varepsilon(y)) \\ &\le g_{A_i}(x, B_\varepsilon(y)) + H_i^\varepsilon P^x(X_{T_i} \in F_{i-3}) \\ &\quad + (1 - c2^i) H_i^\varepsilon P^x(X_{T_i} \in F_{i+4}) \\ &\le \omega_d \varepsilon^d G_i + H_i^\varepsilon (1 - c2^i). \end{aligned} \tag{4.18}$$

Here ω_d denotes the Lebesgue measure of the unit ball.

Holding i fixed but taking the sup over $y \in V_{i+1} - V_i$, $x \in F_{i-2} \cup F_{i+3}$,

$$H_i^\varepsilon \le \omega_d \varepsilon^d G_i + H_i^\varepsilon(1 - c2^i),$$

or, from (4.14),

$$H_i^\varepsilon \le c2^{-i} G_i \omega_d \varepsilon^d \le c2^{i(2-d-\alpha)/\alpha} \omega_d \varepsilon^d. \tag{4.19}$$

Since $\sigma_{i_2} > \tau_1$ and $|y|^\alpha \le cu(y) \le c2^{i+1}$ if $y \in V_{i+1} - V_i$, the strong Markov property yields

$$\int_{B_\varepsilon(y)} \Gamma(z)\,dz \le g(0, B_\varepsilon(y)) \le H_i^\varepsilon \le c|y|^{2-d-\alpha} \omega_d \varepsilon^d. \tag{4.20}$$

Since ε can be arbitrarily small, this yields the upper bound.

Choose i_1 so that $V_{i_1+5} \subseteq B_1$. Replacing i_2 by i_1, replacing sup by inf in the definitions of G_i and H_i^ε, noting $G_i > 0$, and reversing the inequalities in the above argument gives the lower bound. However, the argument is valid only for $y \in V_{i_1}$, which is why the lower bound holds only for a neighborhood of 0. □

5. Existence—nonpolar case

In this section we establish the existence part of Theorem 2.3 in the case $\overline{\mu} < 0$. Suppose f is bounded and continuous on ∂B_1.

Let

$$h(x) = E^x f(X_{\tau_1}). \tag{5.1}$$

Recall the definition of V_i, F_i, and σ_i in (4.7)–(4.9). Write $\operatorname{Osc}_A h$ for $\sup_A h - \inf_A h$.

Proposition 5.1. *There exists $\gamma < 1$ depending only on κ such that*

$$\operatorname*{Osc}_{V_j} h \le \tfrac{1}{2}\gamma \operatorname*{Osc}_{V_{j+1}} h, \quad \textit{provided } V_{j+1} \subseteq B_1.$$

Proof. By taking a linear transformation, we may suppose that $\inf_{V_{j+1}} h = 0$ and $\sup_{V_{j+1}} h = 1$. Since h is continuous in $\overline{V}_{j+1}$ by Lemma 2.6 and $h(x) = E^x h(X_{\sigma_{j+1}})$ for $x \in V_{j+1}$ by the strong Markov property, we have $\inf_{F_{j+1}} h = 0$, $\sup_{F_{j+1}} h = 1$.

Since $h(x) = E^x h(X_{\sigma_j})$ for $x \in V_j$, $\operatorname{Osc}_{V_j} h \le \operatorname{Osc}_{F_j} h$, and the proposition will be proved if we show

$$\operatorname*{Osc}_{F_j} h \le \tfrac{1}{2}\gamma \operatorname*{Osc}_{F_{j+1}} h. \tag{5.2}$$

Fix j_0. In view of Corollaries 4.2 and 4.4 and the definition of u, the minimum distance between F_{j_0} and F_{j_0+1} is bounded below by a constant depending only on κ. So by the Harnack inequality of Krylov-Safonov [9], there exists θ depending only on κ such that for $j = j_0$,

$$\theta \le \frac{P^x(X_{\sigma_{j+1}} \in A\,;\, \sigma_{j+1} < \tau_0)}{P^y(X_{\sigma_{j+1}} \in A\,;\, \sigma_{j+1} < \tau_0)} \le \theta^{-1}, \qquad x, y \in F_j,\ A \subseteq F_{j+1}. \tag{5.3}$$

By scaling, (5.3) holds for all j, with θ independent of j.

Fix $x_0 \in F_j$. By looking at $1 - h$ if necessary, we may suppose

$$P^{x_0}(X_{\sigma_{j+1}} \in A^+\,;\, \sigma_{j+1} < \tau_0) \ge P^{x_0}(X_{\sigma_{j+1}} \in A^-\,;\, \sigma_{j+1} < \tau_0), \tag{5.4}$$

where

$$A^+ = \{y \in F_{j+1} : h(y) \ge \tfrac{1}{2}\}, \qquad A^- = \{y \in F_{j+1} : h(y) \le \tfrac{1}{2}\}.$$

For any $x \in F_j$, $P^x(\tau_0 < \sigma_{j+1}) = P^x(u(X_t)$ hits 0 before hitting $2^{j+1}) = \frac{1}{2}$. So

$$\begin{aligned} h(x) &= E^x h(X_{\sigma_{j+1}}) = E^x(h(X_{\sigma_{j+1}\wedge\tau_0})\,;\, \sigma_{j+1} < \tau_0) + h(0)P^x(\tau_0 < \sigma_{j+1}) \\ &\le P^x(\sigma_{j+1} < \tau_0) + h(0)P^x(\tau_0 < \sigma_{j+1}) \le \tfrac{1}{2} + \tfrac{1}{2}h(0). \end{aligned} \tag{5.5}$$

But also,

$$\begin{aligned} h(x) &= E^x(h(X_{\sigma_{j+1}})\,;\, \sigma_{j+1} < \tau_0) + \tfrac{1}{2}h(0) \\ &\ge \tfrac{1}{2}P^x(X_{\sigma_{j+1}} \in A^+\,;\, \sigma_{j+1} < \tau_0) + \tfrac{1}{2}h(0) \\ &\ge \tfrac{1}{2}\theta P^{x_0}(X_{\sigma_{j+1}} \in A^+\,;\, \sigma_{j+1} < \tau_0) + \tfrac{1}{2}h(0) \\ &\ge \tfrac{1}{4}\theta P^{x_0}(\sigma_{j+1} < \tau_0) + \tfrac{1}{2}h(0) \\ &= \tfrac{1}{8}\theta + \tfrac{1}{2}h(0). \end{aligned} \tag{5.6}$$

Comparing (5.5) and (5.6), we have

$$\operatorname*{Osc}_{F_j} h \le \tfrac{1}{2} - \tfrac{1}{8}\theta = \tfrac{1}{2}(1 - \tfrac{1}{4}\theta),$$

which gives (5.2) with $\gamma = 1 - \frac{1}{4}\theta$. □

Proposition 5.3. *There exists $\delta_1 > 0$ depending only on κ such that*

$$\operatorname*{Osc}_{B_r} h \le c r^{\alpha(1+\delta_1)} \|f\|, \qquad r < 1.$$

Proof. Pick i_1 so that $V_{i_1} \subseteq B_1$. Suppose r is sufficiently small of that $B_r \subseteq V_{i_1}$. Given r, let j be the smallest integer such that $B_r \subseteq V_j$. Then $2^j \le c|r|^\alpha$ and

$$\begin{aligned} \operatorname*{Osc}_{B_r} h &\le \operatorname*{Osc}_{V_j} h \le (\tfrac{1}{2}\gamma)^{i_1 - j} \operatorname*{Osc}_{V_{i_1}} h \le c(\tfrac{1}{2}\gamma)^{-j} \operatorname*{Osc}_{B_1} h \\ &\le c r^{\alpha(1+\delta_1)} \operatorname*{Osc}_{B_1} h, \end{aligned}$$

where $\delta_1 = -\ln\gamma/\ln 2$.

Since $\operatorname{Osc}_{B_1} h \leq 2\|f\|$ follows from (5.1), this proves the proposition for r sufficiently small. Since $\operatorname{Osc}_{B_r} h \leq \operatorname{Osc}_{B_1} h \leq 2\|f\|$, by taking c larger if necessary, we have the proposition for all $r < 1$. □

Proposition 5.3. *There exists $\delta_2 > 0$ depending only on κ such that*

$$\operatorname*{Osc}_{B_r} h \leq c r^{\delta_2} \|f\|, \qquad r < 1.$$

Proof. This is similar to the preceding, but a little simpler. By the Harnack inequality, pick θ' such that

$$\theta' \leq \frac{P^x(X_{\tau_1} \in A)}{P^y(X_{\tau_1} \in A)} \leq (\theta')^{-1}, \qquad x, y \in \partial B_{1/2}, \ A \subseteq \partial B_1. \tag{5.7}$$

Fix $x_0 \in \partial B_r$, $r \leq \frac{1}{2}$, and suppose $\sup_{\partial B_{2r}} h = 1$, $\inf_{\partial B_{2r}} h = 0$, and

$$P^{x_0}(X_{\tau_{2r}} \in A^+) \geq P^{x_0}(X_{\tau_{2r}} \in A^-)$$

with

$$A^+ = \{y \in \partial B_{2r} \colon h(y) \geq \tfrac{1}{2}\}, \qquad A^- = \{y \in \partial B_{2r} \colon h(y) \leq \tfrac{1}{2}\}.$$

Then if $x \in \partial B_r$, $h(x) \leq 1$, and by (5.7) and scaling,

$$\begin{aligned} h(x) = E^x h(X_{\tau_{2r}}) &\geq \tfrac{1}{2} P^x(X_{\tau_{2r}} \in A^+) \\ &\geq \tfrac{1}{2}\theta' P^{x_0}(X_{\tau_{2r}} \in A^+) \geq \tfrac{1}{4}\theta'. \end{aligned}$$

Hence

$$\operatorname*{Osc}_{B_r} h \leq (1 - \tfrac{1}{4}\theta') \operatorname*{Osc}_{B_{2r}} h, \qquad r \leq \tfrac{1}{2}. \tag{5.8}$$

Proposition 5.3 follows from (5.8) similarly to the proof of Proposition 5.2. □

Corollary 5.4. *There exists δ depending only on κ such that*

$$\operatorname*{Osc}_{B_r} h \leq c r^{\alpha+\delta} \|f\|.$$

Proof. Let $\delta = (1 \wedge \delta_1)\delta_2/2$. If $\alpha < \delta_2/2$, the corollary follows by Proposition 5.3. If $\alpha \geq \delta_2/2$, it follows by Proposition 5.2. □

Theorem 5.5. *$h(x)$ defined by* (5.1) *satisfies* (2.8)(i)–(v).

Proof. The boundedness of h is clear by (5.1). Assertions (ii) and (iii) follow by Lemma 2.7. Since h agrees with the unique solution to the Dirichlet problem for L on $B_1 - B_{1/2}$ with boundary function f on ∂B_1 and h on $\partial B_{1/2}$, h is continuous at ∂B_1 and agrees with f there (cf. proof of Lemma 2.7). The continuity of h at 0 comes from Lemma 2.6, and it remains to prove (v).

By the bounds on Γ from Theorem 4.7 and the fact that $h \in C^2$ on $B_1 - \{0\}$, it suffices to restrict attention to $R < 1/2$. By adding or subtracting a constant, we may suppose $h(0) = 0$. Then if $x \in B_R$, h solves the Dirichlet problem for

L on the ball $B_{|x|/2}(x)$ with boundary function h by Lemma 2.7. So $h \in C^2$ there, and it is well known [5, Chapter 6] that

$$(5.9) \qquad |D_{ij}h(y)| \le c|x|^{-2} \sup_{z\in\partial B_{|x|/2}(x)} |h(z)|, \qquad y \in B_{|x|/4}(x).$$

The constant c may depend on the smoothness of the a_{ij}.

But then

$$(5.10) \qquad |D_{ij}h(x)| \le c|x|^{-2} \sup_{z\in B_{3|x|/2}} |h(x)| \le c|x|^{-2} \operatorname*{Osc}_{B_{3|x|/2}} h \le c|x|^{-2+\alpha+\delta},$$

by Corollary 5.4 and the fact that $h(0) = 0$.

Estimate (5.10) and Theorem 4.7 give, changing to polar coordinates,

$$\int_{B_R} |D_{ij}h(x)|^{1+\eta}\Gamma(x)\,dx \le c\int_0^R r^{d-1}(r^{-2+\alpha+\delta})^{1+\eta} r^{2-d-\alpha}\,dr.$$

Recalling Lemma 4.6, this gives (v) if $\eta < \delta/(2-\alpha-\delta)$. □

Corollary 5.6. *There exists $\varepsilon > 0$ depending only on κ such that*

$$\nabla h \in L^{d+\varepsilon}(B_R), \qquad D_{ij}h \in L^{d/2+\varepsilon}(B_R), \qquad R < 1.$$

Proof. The second assertion follows by integrating estimate (5.10) in polar coordinates. Just as we obtained (5.10), we get

$$(5.11) \qquad |D_i h(x)| \le c|x|^{-1} \sup_{z\in B_{3|x|/2}} |h(z)| \le c|x|^{-1+\alpha+\delta}.$$

Integrate (5.11) in polar coordinates to get the first assertion. □

6. Uniqueness—nonpolar case

In this section we complete the proof of Theorem 2.3 for the case $\overline{\mu} < 0$ by showing uniqueness. We first show uniqueness for the Dirichlet problem for V_{i_1}, where i_1 is chosen so that $V_{i_1} \subset B_1$.

Proposition 6.1. *Suppose f is a bounded continuous function on F_{i_1}. Suppose v_1 and v_2 are two bounded continuous functions on V_{i_1}, agreeing with f on F_{i_1}, C^2 in $V_{i_1} - \{0\}$, satisfying $Lv_j = 0$ on $V_{i_1} - \{0\}$, $j = 1, 2$, and satisfying hypothesis* (2.8)(v) *for some $R < 1$. Then $v_1 = v_2$ on V_{i_1}.*

Proof. By considering $v = v_1 - v_2$, we may suppose $f \equiv 0$. Since L has smooth coefficients on $B_1 - \{0\}$, if $v(0) = 0$, then by the usual maximum principle, $v \equiv 0$ on V_{i_1}. We suppose then that $v(0) \ne 0$ and obtain a contradiction. By multiplying by a constant, we may suppose $v(0) = 1$.

Consider $w(x) = P^x(\tau_0 < \sigma_{i_1})$. By Proposition 2.2, proved in §3, $v - w$ is bounded and continuous on V_{i_1}, $(v - w)(0) = 0$, $v - w = 0$ on F_{i_1}, $v - w$ is C^2 on $V_{i_1} - \{0\}$, and $L(v - w) = 0$ there. By the usual maximum principle, $v = w$ on V_{i_1}.

If $x = (r, \theta) \in V_{i_1}$ is such that $u(x) = \beta$, then

$$(6.1)\quad \begin{aligned} w(x) &= P^x(\tau_0 < \sigma_{i_1}) = P^x(u(X_t) \text{ hits } 0 \text{ before hitting } 2^{i_1}) = \frac{2^{i_1} - u(x)}{2^{i_1}} \\ &= 1 - \frac{\beta}{2^{i_1}} = 1 - c\beta = 1 - cu(x) = 1 - cr^\alpha d(\theta). \end{aligned}$$

But then $D_{rr}w(x) = c|x|^{\alpha-2} d(\theta) \geq c|x|^{\alpha-2}$. And

$$\int_{B_R} \sum_{i,j=1}^{d} |D_{ij}w|^{1+\eta}(x)\Gamma(x)\,dx \geq c\int_0^R r^{d-1}(r^{\alpha-2})^{1+\eta} r^{2-\alpha-d}\,dr = \infty,$$

since by Lemma 4.6, $\alpha \leq 1$, hence $d - 1 + (1+\eta)(\alpha - 2) + 2 - \alpha - d < -1$. This contradicts the assertion that $v = w$ satisfied (2.8). Hence the proof is complete. □

We now complete the proof of Theorem 2.3 in the case $\overline{\mu} < 0$.

Theorem 6.2. *Suppose* $\eta > 0$. *There is at most one function* h *satisfying hypotheses* (2.8)(i)–(v).

Proof. Just as in Proposition 6.1, the theorem will be proved if we show $w(x) = P^x(\tau_0 < \tau_1)$ does not satisfy (2.8)(v).

Consider the Dirichlet problem on V_{i_1} with boundary function w. Let $h(x) = E^x w(X_{\sigma_{i_1}})$. If r_0 is taken small enough so that $B_{r_0} \subseteq V_{i_1}$, then by the strong Markov property $h(x) = E^x h(X_{\tau_{r_0}})$. By applying scaling to Theorem 5.5, h satisfies (2.8)(i)–(v) if B_1 is replaced by B_{r_0}. Also, since h solves the Dirichlet problem on $V_{i_1} - B_{r_0}$ with boundary function w on F_{i_1}, h on ∂B_{r_0}, h is one solution to the Dirichlet problem on V_{i_1} satisfying the hypotheses of Proposition 6.1. Moreover, by Corollary 5.4, there exists $\delta > 0$ such that

$$(6.2)\qquad \operatorname*{Osc}_{B_r} h \leq cr^{\alpha+\delta}.$$

Suppose now that $w(x)$ satisfies (2.8)(v). By the proof of Proposition 2.2, w also satisfies the hypotheses of Proposition 6.1. Therefore, by the conclusion of Proposition 6.1, $w = h$ on V_{i_1}.

But by Proposition 4.5, we know $1 - w(x) \geq c|x|^\alpha$ for x small. Since $w(0) = 1$, this contradicts (6.2). Therefore w cannot satisfy (2.8)(v), which proves the theorem. □

7. Polar, transient case

In this section we prove Theorem 2.3 in the case $\overline{\mu} > 0$. The proof is considerably easier in this case.

For $b > 1$, define

$$(7.1)\qquad \begin{gathered} Q_b f(x) = E^{bx}(f(X_{\tau_1}); \tau_1 < \infty), \\ Q = Q_2. \end{gathered}$$

Completely analogously to the $\overline{\mu} < 0$ case, Q_b is strongly positive and compact. The eigenfunction $d(\theta)$ corresponding to the largest eigenvalue of Q is strictly positive. As in §4, there exists $\alpha > 0$ such that

$$u(x) = E^x(d(X_{\tau_1}); \tau_1 < \infty) = r^{-\alpha} d(\theta), \qquad x = (r, \theta), \ r > 1, \tag{7.2}$$

is C^∞ and solves $Lu = 0$ in $\overline{B}_1^c$, but now note the exponent of r in (7.2) is negative. Arguing as in (4.6),

$$-\alpha(-\alpha - 1)d/2 - \alpha(2\mu + 1)d/2 + L_S d - \alpha M(rd) = 0$$

for all θ, hence $Lu(x) = 0$ for all $x \neq 0$. Therefore $u(X_{t\wedge\tau_0}) = u(X_t)$ (since $\tau_0 = \infty$, a.s.) is a local martingale.

Let

$$V_i = \{x : u(x) \geq 2^{-i}\}, \qquad i = \dots, -2, -1, 0, 1, 2, \dots, \tag{7.3}$$

so that we still have $V_i \subseteq V_{i+1}$. Define F_i and σ_i as in (4.8), (4.9).

Theorem 7.1. *There exists a constant c (depending on α) such that*

$$\Gamma(x) \leq c|x|^{2-d} \quad \text{a.e. on } B_1.$$

Proof. The proof is very similar to that of Theorem 4.7, with one exception. If $z \in F_{i+4}$, $i \leq i_2 - 5$,

$$\begin{aligned} P^z(\sigma_{i+3} < \sigma_{i_2}) &= P^z(u(X_t) \text{ hits } 2^{-(i+3)} \text{ before hitting } 2^{-i_2}) \\ &= \frac{2^{-(i+3)} - 2^{-(i+4)}}{2^{-(i+3)} - 2^{-i_2}} \leq 1 - c, \qquad c > 0 \end{aligned} \tag{7.4}$$

(cf. with $1 - c2^i$ in (4.16)). Just as in (4.17), $P^x(X_{T_i} \in F_{i+4})$ is still bounded above and below by constants independent of i.

With (7.4), (4.19) becomes

$$H_i^\varepsilon \leq cG_i\omega_d\varepsilon^d \leq c2^{i(2-d)/\alpha}\omega_d\varepsilon^d \tag{7.5}$$

and (4.20) becomes

$$\int_{B_\varepsilon(\eta)} \Gamma(z)\,dz \leq c|y|^{2-d}\omega_d\varepsilon^d,$$

which yields the desired result. □

Note that Proposition 5.3 makes no use of the hypothesis $\overline{\mu} < 0$, and hence the assertion is valid in the cases $\overline{\mu} \geq 0$.

Suppose f is bounded and continuous on ∂B_1. Let

$$h(x) = E^x f(X_{\tau_1}). \tag{7.6}$$

Theorem 7.2. *$h(x)$ defined by* (7.6) *satisfies* (2.8)(i)–(v). *There is at most one function h satisfying* (2.8)(i)–(v).

Proof. Following the proof of Theorem 5.5, (2.8)(i)–(iv) hold. We may suppose $h(0) = 0$ and $R < 1/2$, and then exactly as in the derivation of (5.10),

$$|D_{ij}h(x)| \leq c|x|^{-2} \operatorname*{Osc}_{B_{3|x|/2}} h. \tag{7.7}$$

Applying Proposition 5.3,

$$|D_{ij}h(x)| \leq c|x|^{-2+\delta_2}\|f\|, \quad |x| < 1/2. \tag{7.8}$$

Therefore, switching to polar coordinates,

$$\int_{B_P} |D_{ij}h(x)|^{1+\eta}\Gamma(x)\,dx \leq c\int_0^R r^{d-1}(r^{-2+\delta_2})^{1+\eta}r^{2-d}\,dr < \infty, \tag{7.9}$$

provided $\eta < \delta_2/(2-\delta_2)$.

For the uniqueness assertion, apply the extended maximum principle Theorem 2.1 to the difference of any two functions h_1, h_2. Then $h_1 - h_2 = 0$ on $B_1 - \{0\}$, and by continuity, $h_1(0) = h_2(0)$. □

Corollary 7.3. *Corollary* 5.6 *holds in the cases* $\overline{\mu} \geq 0$.

Proof. Use the estimate (7.8) and

$$|D_i h(x)| \leq c|x|^{-1} \sup_{x\in B_{3|x|/2}} |h(z)| \leq C|x|^{-1+\delta_2}$$

in place of (5.10) and (5.11) in the proof of Corollary 5.6. □

8. Polar, recurrent case

It remains to prove Theorem 2.3 in the case $\overline{\mu} = 0$.

Define d and u by (3.7) and (3.9). Let

$$V_i = \{y = (r, \theta): r \leq 2^i e^{-d(\theta)}\}, \qquad i = \ldots, -2, -1, 0, 1, 2, \ldots, \tag{8.1}$$

so that $V_i \subseteq V_{i+1}$, and if $y \in F_i = \partial V_i$, $u(y) = i\log 2$.

Theorem 8.1. *There exists a constant* c *such that*

$$\Gamma(x) \leq c(1 + |\log x|)|x|^{2-d} \quad \textit{a.e. on } B_1.$$

Proof. As in the proof of Theorem 7.1, we modify the proof of Theorem 4.7 by writing: if $z \in F_{i+4}$, $i \leq i_2 - 5$,

$$\begin{aligned} P^z(\sigma_{i+3} < \sigma_{i_2}) &= P^z(u(X_t) \text{ hits } (i+3)\log 2 \text{ before hitting } i_2\log 2) \\ &= \frac{i_2 - (i+4)}{i_2 - (i+3)} \leq 1 - c(1+|i|)^{-1}, \end{aligned} \tag{8.2}$$

leading to

$$H_i^\varepsilon \leq c(1+|i|)G_i\omega_d\varepsilon^d \tag{8.3}$$

and

$$\int_{B_\varepsilon(y)} \Gamma(z)\,dz \leq c(1 + |\log y|)|y|^{2-d}\omega_d\varepsilon^d. \quad □ \tag{8.4}$$

Let f be bounded and continuous on ∂B_1,

$$h(x) = E^x f(X_{\tau_1}). \tag{8.5}$$

Theorem 8.2. *$h(x)$ defined by* (8.5) *satisfies* (2.8)(i)–(v). *There is at most one function h satisfying* (2.8)(i)–(v).

Proof. The proof is identical to that of Theorem 7.2, except (7.9) becomes

$$(8.6)\qquad \int_{B_R} |D_{ij}h(x)|^{1+\eta}\Gamma(x)\,dx \le c\int_0^R r^{d-1}(r^{-2+\delta_2})^{1+\eta}|\log r| r^{2-d}\,dr < \infty,$$

provided $\eta < \delta_2/(2-\delta_2)$. □

Note that Corollary 7.3 includes the case $\overline{\mu} = 0$ in its statement.

REFERENCES

1. R. F. Bass and E. Pardoux, *Uniqueness for diffusions with piecewise constant coefficients*, Probability Theory and Related Fields **76** (1987), 557–572.
2. J. R. Baxter and G. A. Brosamler, *Energy and the law of the iterated logarithm*, Math. Scand. **38** (1976), 115–136.
3. R. N. Bhattacharya, *On the functional central limit theorem and the law of the iterated logarithm for Markov processes*, Z. Wahrsch. Verw. Gebiete **60** (1982), 185–201.
4. D. Gilbarg and J. Serrin, *On isolated singularities of solutions of second order elliptic differential equations*, J. Analyse Math. **4** (1955/56), 309–340.
5. D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Springer, New York, 1983.
6. M. G. Krein and M. A. Rutman, *Linear operators leaving invariant a cone in a Banach space*, Amer. Math. Soc. Transl. **10** (1962), 199–325.
7. N. V. Krylov, *An inequality in the theory of stochastic integrals*, Theory Probab. Appl. **16** (1971), 438–448.
8. N. V. Krylov and M. V. Safonov, *An estimate of the probability that a diffusion process hits a set of positive measure*, Soviet Math. Dokl. **20** (1979), 253–255.
9. ——, *A certain property of solutions of parabolic equation with measurable coefficients*, Math. USSR-Izv. **16** (1981), 151–235.
10. L. Lamberti and P. Manselli, *Existence-uniqueness theorems and counterexamples for an axially symmetric elliptic operator*, Boll. Un. Ital. B **2** (1983), 431–443.
11. C. Pucci, *Limitazioni per soluzioni di equazioni ellittiche*, Ann. Mat. Pura Appl. (4) **74** (1966), 15–30.
12. M. V. Safonov, *Unimprovability of estimates of Hölder continuity for solution to linear elliptic equations with measurable coefficients*, Mat. Sb. **132 (174)** (1987), 275–288. (Russian)
13. D. W. Stroock and S. R. S. Varadhan, *Multidimensional diffusion processes*, Springer, New York, 1979.
14. R. J. Williams, *Brownian motion with polar drift*, Trans. Amer. Math. Soc. **292** (1985), 225–246.
15. L. Caffarelli, private communication.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WASHINGTON 98195